# An Analytic Second-Order Dividend Correction for Variance Swap Contracts on Stock Indices

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## 1 Abstract

We present an analytic dividend-payout modification to the valuation of a typical equity index variance swap contract. Including these dividends in the valuation reveals the risk that the contracts have to changes in the expected future dividends or, more precisely, to market dividend swap rates. This adjustment is also necessary from a calibration point of view because the volatility curve "basis" in lognormal stock price change models is usually the dividend-free volatility curve, $\sigma(t)$, that is used in the lognormal process

$$dS = S\mu(t)\,dt + S\sigma(t)\,dz \quad . \tag{1}$$

Known future dividends are typically introduced as instantaneous downward jumps in the paths of future stock prices, independently of the above process. Assuming piecewise lognormal index-level evolution, we display the effect of future dividends on the the index variance swap contract to second order in dividends as an analytic expression. Moreover, we demonstrate that this term must be at least second order, ie one cannot generally go to only first order in dividends when assuming piecewise lognormality. The resultant dividend risk is found to be typically quite small compared to the cash value at risk in common contracts. For short-tenored contracts, however, say one month, this dividend adjustment is arguably not negligible, particularly in view of current bid-offer spreads on these contracts.

## 2 Introduction

In general practise, there are two standard types of variance swap contracts (VSCs): (1) the individual stock VSC, which explicitly adjusts for dividends in the calculation of variance of returns; and (2) the index VSC which does *not* explicitly adjust for dividends. The object of this note is to outline how to correctly incorporate dividends in the valuation of these contracts, and, for the simple case of piecewise lognormal stock evolution, present an analytic modification to the formula for index VSC valuation.

During the observation period of the individual stock VSC, past, realised dividend payouts are accounted for by appropriately adjusting the stock price's closing value on the ex-dividend trading day. Thus, the "synthetic variance" caused by a non-stochastic stock price jump after a dividend payout – the dividend jump – is adjusted away. Known future, unrealised dividends are simply ignored in the valuation of the future expected payoff. This is equivalent to valuing the stock VSC in the ex-dividend basis.

By contrast, the typical index VSC has no explicit adjustment for dividend payouts in the contract specifications. Thus, every dividend jump will affect the cumulative realised variance. The past realised cumulative variance will include the extra "synthetic variance" caused by dividend jumps because the past realised closing prices are *not* dividend-adjusted. The issue then is to determine by how much the future expected payoff is affected by known future unrealised dividends, relative to a volatility that is "ex-dividends".

To put it another way, the valuation of a VSC halfway through the observation period has two essential terms: one for realised variance of past price movements and another for expected future price movements. For the valuation of the individual stock VSC, one adjusts away the effect of dividends by modifying the former term. Conversely, the valuation of the index VSC requires a modification to the latter term in order to capture dividend effects.

In the following note, we will use a piecewise lognormal index process to approximate the future expected payoff of an index VSC to second order in known, fixed dividends. (A generalisation to dividends of known proportion is possible and straight forward.) We assume that values of the dividends are small relative to the index level. To simplify the results, we will model a forward VSC – we will not calculate the realised and intra-day variance. Without loss of generality, we will assume that the input index volatility and drift rate are both constant and that the index has no borrow fee.

Using these assumptions, we analyse the effect of fixed dividends on a one-month forward VSC on the EURO STOXX 50. The correction to the forward realised volatility, at today's prices of about 24 volatility points, is approximately 0.13 volatility points.

## 2.1 The Contract

Consider the following typical index VSC:

$$\text{Payment at Settlement} = \frac{\text{Vega Amount}}{2 \times \text{strike}} \times \left(FRV^2 - \text{strike}^2\right) \quad , \tag{2}$$

$$FRV = 100 \times \sqrt{\frac{252}{N} \sum_{i=1}^{N} \left[\ln\left(\frac{P_i}{P_{i-1}}\right)\right]^2} \quad , \tag{3}$$

where $FRV$ is the final realised volatility, $N$ is the number of observation days of the contract ($\frac{N}{252}$ is the number of years), "strike" is the strike volatility, and $P_i$ is the closing index price on the observation day $i$. The number of contract units, $\frac{\text{Vega Amount}}{2\times\text{strike}}$, is a typical position convention: when $FRV^2 \approx \text{strike}^2$, this convention allows the trader to think in terms of the vega-position of a *volatility* swap contract.

One important feature of this contract definition is that the "variance" defined by the $FRV$ is actually the second moment of the index's return distribution since it does not adjust away the squared mean of the return. This quality of the definition enhances the effect of dividends on the contract valuation.

We now focus on how to value such contracts when unrealised dividends are known. Valuing the contract one day before its start date, the objective is to write the risk-free expected value of the $FRV^2$,

$$\left(FRV^2\right)_{\text{exp}} = 100^2 \frac{252}{N} \sum_{i=0}^{N} \left\langle \left[\ln\left(\frac{P_i}{P_{i-1}}\right)\right]^2 \right\rangle_{\text{risk-free}} \quad , \tag{4}$$

in terms of the input lognormal volatility function, $\sigma(t)$, and the known future dividends.

## 3 Valuation of the Forward Contract with Dividends

In this section, we demonstrate the modification to the index VSC caused by $n$ unrealised but known dividends. We value the VSC forward one day before its start date so that there is no realised variance. As previously stated, we use a piecewise lognormal index process and, for simplicity, assume that index has a constant volatility, constant drift, and no borrow fee. We also assume that the riskless interest rate is constant. The generalisation to include dividends of known proportion of the index level, as well as term structured non-stochastic volatility, riskless rates, and borrow rates, is straightforward.

When there are no dividends, the lognormal process at instant $t$ for the index level $P_t$ is given by

$$\frac{dP_t}{P_t} = \mu dt + \sigma dz \quad , \tag{5}$$

where

$$z = \text{Wiener process} \quad , \quad \mu = \text{index drift} \quad , \quad \sigma = \text{index volatility} \quad . \tag{6}$$

Accordingly, there exists a Gaussian state variable $x_t$ such that

$$dx_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz \quad , \tag{7}$$

$$\Delta x_{T,t} = \int_{u=t}^{u=T} dx_u \quad , \quad P_T = P_t e^{\Delta x_{T,t}} \quad , \tag{8}$$

for $T \geq t$.

Suppose that for two known but unrealised fixed dividends, $D_1$ and $D_2$, we have the respective ex-dividend days $t_1$ and $t_2$, with $t_2 \geq t_1$. Assuming a piecewise lognormal index evolution, ie a lognormal evolution with non-stochastic ex-dividend jumps, it can be shown that for $T \geq t_2$,

$$P_T = P_t e^{\Delta x_{T,t}} \left[ 1 - \sum_{k=1}^{n=2} \frac{D_k}{P_t} e^{-\Delta x_{T,t_k}} \right] \quad . \tag{9}$$

For ease of notation, denote the discrete closing values of the index by $i$:

$$P_{t=\text{close day } i} \equiv P_i \quad . \tag{10}$$

For the present forward evaluation, we then have

$$P_{t=\text{close today}} \equiv P_0 \quad , \quad P_{t=\text{close tomorrow}} \equiv P_1 \quad , \quad \text{etc} \quad . \tag{11}$$

In order to calculate the final realised volatility of Eq (4), we insert Eq (9) into it (with $n$ now arbitrary), expand about small dividends relative to the index level, and perform the expectation value. With tomorrow as the first observation day of the contract, the forward value at today's close is

$$\left(FRV^2\right)_{\text{exp}} \approx \left(FRV^2\right)_{\text{exp}}^{\text{no divs}} + 100^2 \frac{252}{N} \sum_{k=1}^{n} \Phi_k \quad , \tag{12}$$

where the correction terms $\Phi_k$ are labelled by the ex-dividend dates and

$$t_k = \frac{1}{365} \times \left( \text{days from today's close to close of } k^{\text{th}} \text{ ex-dividend date} \right) \quad . \tag{13}$$

(We are using the convention of measuring drift and variance in the "actual-over-365 basis".) The correction terms are found to be

$$\begin{aligned} \Phi_k = &-\frac{2}{365} \left( \frac{D_k}{P_0} \right) \left( \mu - \frac{\sigma^2}{2} \right) e^{\left(\sigma^2 - \mu\right)\left(t_k - \frac{1}{365}\right)} \left[ 1 + \sum_{m=1}^{k-1} \frac{D_m}{P_0} e^{\left(2\sigma^2 - \mu\right)\left(t_m - \frac{1}{365}\right)} \right] \\ &+ \left( \frac{D_k}{P_0} \right)^2 \left[ 1 - \frac{1}{365} \left( \mu - \frac{\sigma^2}{2} \right) \right] e^{\left(3\sigma^2 - 2\mu\right)\left(t_k - \frac{1}{365}\right)} \quad , \end{aligned} \tag{14}$$

and the dividend-free term is

$$\left(FRV^2\right)_{\text{exp}}^{\text{no divs}} = 100^2 \frac{252}{N} \left[ \sigma^2 T_{\text{mat}} + \left( \mu - \frac{1}{2}\sigma^2 \right)^2 T_{\text{mat}}^2 \right] \quad , \tag{15}$$

where $T_{\mathrm{mat}}$ is the maturity date of the VSC in the actual-over-365 basis. As a final step, value the above risklessly, ie

$$\mu = r \quad , \tag{16}$$

for the constant, risk-free cash interest rate $r$. This gives the contract's price in the standard risk-neutral pricing paradigm.

Note that for qualitatively reasonable inputs, say $\frac{D}{P} \sim 1.0 \times 10^{-4}$, $\sigma \sim 0.20$, and $\mu \sim 0.05$, we see that the first piece of second-order term in Eq (14), which is essentially $\left(\frac{D}{P}\right)^2 \sim 1.0 \times 10^{-8}$, is of the same order of magnitude as the first order term. Thus, when using piecewise lognormal evolution to find adjustments for dividend jumps, it is necessary to go to second order in dividends.

Also note that the second term of Eq (15) arises from the fact the the "variance" defined in the VSC is actually the second moment of the Gaussian distribution over the state variable $x_t$. If the VSC were defined to remove the squared drift, then this term would not appear. Removing the squared drift also eliminates most of the dividend correction terms of Eq (14). The remaining correction terms would be second order in dividends but would have small coefficients of order $[\exp(\sigma^2 \Delta t_{k,k-1}) - 1]$. Thus we see that the driftless variance definition of the VSC enhances the effect of dividends. This is an important observation.

# 4 Analysis

In order to analyse our result, we calculated the dividend correction for a one-month forward VSC on the EURO STOXX 50, which has a dividend yield of about 4%. The forward observation time was from 23 April 2008 to 23 May 2008, and the calculations were done at the market's close on 22 April 2008.

During this observation time, 30 of the 50 stocks had scheduled ex-dividend dates. Many of these overlapped, however, so the index had in effect 14 ex-dividend dates. All of the stock dividends had to be weighted properly in order to convert them into index dividends.

This was done by multiplying each stock's dividend by the number of shares of its respective stock in the index and then dividing by the overall index divisor. Finally, dividends on the same day were summed to get the aggregate index dividend for that day.

Using these 14 aggregate dividends, $D_k$, and the inputs

$$\mu = r = 5\% \quad , \quad \sigma = 23.8\% \quad , \tag{17}$$

$$P_0 = \text{index level at close of 22 April 2008} = €3736.12 \quad , \tag{18}$$

in Eqs (12) - (15), we obtained

$$\sqrt{(FRV^2)^{\text{no divs}}_{\text{exp}}} \approx 22.97\% \quad , \tag{19}$$

and

$$\sqrt{(FRV^2)_{\text{exp}}} \approx 23.10\% \quad , \tag{20}$$

for the dividend-free and dividend-adjusted forward-realised volatilities, respectively. Thus, our dividend adjustment was approximately 0.13 volatility points.

# 5 Conclusion

Individual equity variance swap contracts typically carry no dividend risk since they usually adjust away the effects of dividends in their definition of the final realised volatility. Conversely, index VSCs do not adjust away the affects of dividends in their definition. Therefore, these contracts inherently carry dividend risk.

The effect of dividends on a one month VSC on an index that pays a 4% dividend yield can be on the order of magnitude of 10 basis points in volatility, particularly if a large portion of the dividends are paid during the contract, as was the case in our analysis. Considering that typical bid-offer spreads for index VSCs are about 100 basis points, this dividend effect is a small but non-negligible risk over a large portfolio. The risk can be hedged with dividend swaps, or it can at least be calculated to ensure that it is acceptable.

# References

[1] K. Demeterfi, E. Derman, M. Kamal, J. Zou, (1999) "More Than You Ever Wanted to Know About Volatility Swaps", *Quantitative Strategies Research Notes, March 1999*, Goldman, Sachs, & Co.